THE CYCLOID AND THE TAUTOCHRONE PROBLEM

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August 27, 2012

Abstract

In this dissertation we will first introduce historically the invention of the Pendulum by Christiaan Huygens, in particular the cycloidal one. Then we will discuss mathematically the cycloid curve, related to the *Tautochrone Problem* and we'll compare the circular and cycloidal pendulum. Finally we'll introduce Abel's Integral Equation as another way to attack and solve the Tautochrone Problem.

1 Historical Introduction

Christiaan Huygens (14 April 1629 - 8 July 1695), was a prominent Dutch mathematician, astronomer, physicist and horologist. His work included early telescopic studies elucidating the nature of the rings of Saturn and the discovery of its moon Titan, the invention of the pendulum clock and other investigations in timekeeping, and studies of both optics and the centrifugal force.
In 1657 patented his invention of the pendulum clock and in 1673 published his mathematical analysis of pendulums, *Horologium Oscillatorium sive de motu pendulorum*, his greatest work on horology. It had been observed by Marin Mersenne and others that pendulums are not quite isochronous, that is, their period depends on their width of swing, wide swings taking longer than narrow swings. Huygens analysed this problem by finding the shape of the curve down which a mass will slide under the influence of gravity in the same amount of time, regardless of its starting point; the so-called

Tautochrone Problem. By geometrical methods which were an early use of calculus, he showed that this curve is a Cycloid.

> "On a cycloid whose axis is erected on the perpendicular and whose vertex is located at the bottom, the times of descent, in which a body arrives at the lowest point at the vertex after having departed from any point on the cycloid, are equal to each other..."[1]

2 The Cycloid

The cycloid is the locus of a point on the rim of a circle of radius R rolling without slipping along a straight line. It was first studied by Nicola Cusano and it was named by Galileo in 1599. It is also the solution to the *Tautochrone Problem.*
The cycloid is represented by a parametric expression that contains the radius R of the circle and the rotation angle θ. The following formula represents two cycloids generated by a circle of radius R, one trough the origin and the other trough π, which consist of the points (x, y), with

$$\mathcal{C}^{\pm} \equiv \begin{cases} x = R(\theta \pm \sin\theta) \\ y = R(1 - \cos\theta) \end{cases} \qquad \theta \in [-\pi, \pi]$$

For a given θ, the circle's centre lies at $x = R\theta$, $y = R$.

2.1 Arc Length

The arc length of a curve defined parametrically by $x = f(t)$ and $y = g(t)$ is

$$s(\theta) = \int_0^\theta ds \qquad \text{where} \qquad ds = \sqrt{dx^2 + dy^2}$$

For the cycloid we have

$$\begin{cases} dx = R(1 \pm \cos\theta)d\theta \\ dy = R\sin\theta d\theta \end{cases}$$

[1]Blackwell, Richard J. (1986). *Christiaan Huygens' The Pendulum Clock.* Ames, Iowa: Iowa State University Press. ISBN 0-8138-0933-9. Part II, Proposition XXV, p. 69

so

$$ds = 2R\cos\frac{\theta}{2}d\theta \quad \Rightarrow \quad s(\theta) = 4R\sin\frac{\theta}{2}$$

2.2 Critical Points

From the study of the first derivative we obtain

$$\frac{dy}{dx} = \frac{\sin\theta}{1\pm\cos\theta} = \begin{cases} \tan\frac{\theta}{2} & (\mathcal{C}^+) \\ \cot\frac{\theta}{2} & (\mathcal{C}^-) \end{cases}$$

we have the maxima for $\mathcal{C}^+$ in $\theta = 0 + k\pi$ and minima for $\mathcal{C}^-$ in $\theta = k\pi$. While from the study of the second derivative we get[2]

$$\frac{d^2y}{dx^2} = \pm\frac{1}{(1\pm\cos\theta)^2} = \begin{cases} \frac{1}{4\cos^2\frac{\theta}{2}} > 0 & (\mathcal{C}^+) \\ \frac{-1}{4\sin^2\frac{\theta}{2}} < 0 & (\mathcal{C}^-) \end{cases}$$

$\mathcal{C}^+$ has always a positive curvature and $\mathcal{C}^-$ a negative one.

$$A' \Leftrightarrow \theta = -\pi \quad \begin{cases} x = -\pi R \\ y = 2R \end{cases} \quad ; \quad O \Leftrightarrow \theta = 0 \quad \begin{cases} x = 0 \\ y = 0 \end{cases} \quad ; \quad A \Leftrightarrow \theta = \pi \quad \begin{cases} x = \pi R \\ y = 2R \end{cases}$$

For $\mathcal{C}^+$ the circle rolls over the line $y = 0$
For $\mathcal{C}^-$ the circle rolls under the line $y = 2R$

[2]Let us recall

$$\begin{cases} \cos^2\alpha - \sin^2\alpha = \cos 2\alpha \\ 2\cos\alpha\sin\alpha = \sin 2\alpha \end{cases} \Rightarrow \begin{cases} \sin^2\alpha = \frac{1-\cos 2\alpha}{2} \\ \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \end{cases}$$

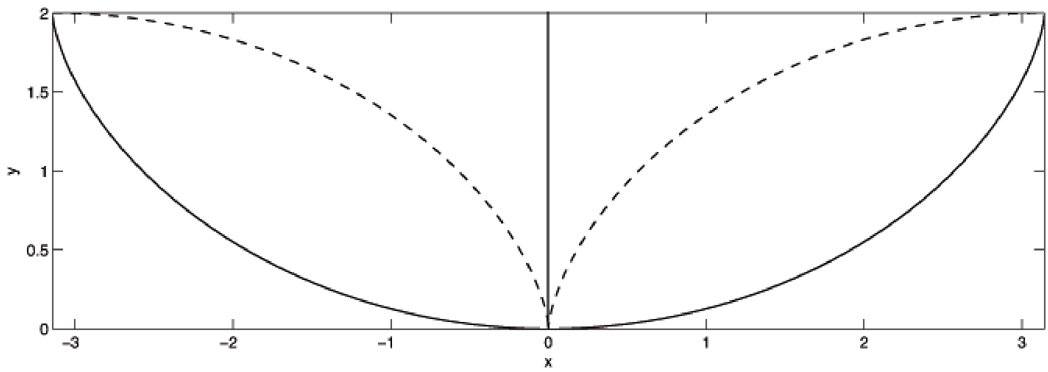

Figure 1: Plot of Cycloid with R = 1 and $-\pi < \theta < \pi$, $\mathcal{C}^+$ (continuous) and $\mathcal{C}^-$ (dashed)

3 Circular Pendulum vs. Cycloidal Pendulum

Now we consider the equation of motion for a particle mass constrained to move in a vertical curve without friction. The equation is:

$$F_t(s) = m\frac{d^2s}{dt^2}$$

where $F_t(s)$ is the tangential force and s the curvilinear abscissa. Gravity is the only force acting upon the mass, so

$$F_t(s) = -mg\cos\phi; \qquad \cos\phi = \frac{dy}{ds}.$$

With this equation one can describe the motion of two type of pendulums: the Circular Pendulum, first studied by Galileo Galilei in 1581, and the Cycloidal Pendulum, studied by Huygens in 1673.

3.1 Cycloidal Pendulum

Starting from the previous equation for a cycloidal curve, generated by a circle of radius R

$$\begin{cases} x = R(\theta \pm \sin\theta) \\ y = R(1-\cos\theta) \end{cases} \qquad -\pi \leq \theta \leq \pi$$

we have

$$dy = R\sin\theta\, d\theta \qquad \text{and} \qquad ds = 2R\cos\frac{\theta}{2}\, d\theta$$

then we obtain

$$\cos\phi = \frac{dy}{ds} = \frac{\sin\theta}{2\cos\frac{\theta}{2}} = \sin\frac{\theta}{2}$$

and the equation of motion becomes

$$\frac{d^2s}{dt^2} + g\sin\frac{\theta}{2} = 0$$

also noting that

$$s(\theta) = 4R\sin\frac{\theta}{2} \Longrightarrow \sin\frac{\theta}{2} = \frac{s}{4R}$$

we finally have

$$\frac{d^2s}{dt^2} + \frac{g}{4R}s = 0$$

The result is an *harmonic motion* $\forall s = s(\theta)$ with period

$$T = 2\pi\sqrt{\frac{4R}{g}}$$

that does NOT depend on the amplitude of the oscillation.

3.2 Circular Pendulum

In the case of a circular curve of radius L

$$\begin{cases} x = L\sin\alpha \\ y = L(1-\cos\alpha) \end{cases} \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$$

we have

$$dy = L\sin\alpha\, d\alpha \qquad \text{and} \qquad ds = L\, d\alpha$$

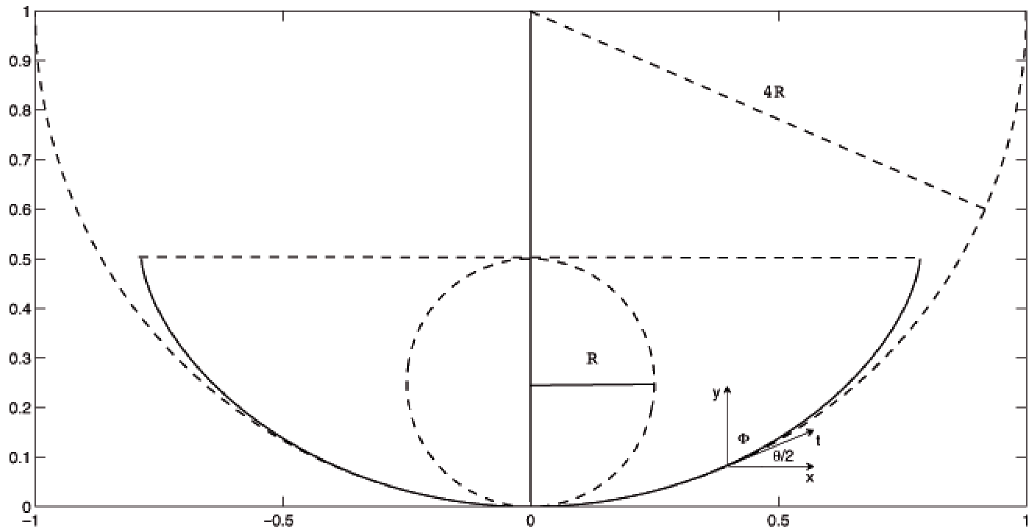

Figure 2: Cycloidal pendulum: $T = T_0$, Circular pendulum $T \geq T_0$, $T_0 = 2\pi\sqrt{\frac{L}{g}}$ with $L = 4R$

then we obtain

$$\cos\phi = \frac{dy}{ds} = \sin\alpha$$

and the equation of motion becomes

$$\frac{d^2 s}{dt^2} + g\sin\alpha = 0$$

$$\frac{d^2\alpha}{dt^2} + \frac{g}{L}\sin\alpha = 0$$

In this case there is an *harmonic motion* only for $\sin\alpha \approx \alpha$ with period

$$T \simeq 2\pi\sqrt{\frac{L}{g}}.$$

4 Abel and Tautochrone

Niels Henrik Abel attacked a generalized version of the *Tautochrone Problem* (Abel's mechanical problem), namely, to find, given a function $T(y)$ that specifies the total time of descent for a given starting height, an equation for the curve that yields such result. The *Tautochrone Problem* is a special case of Abel's mechanical problem when $T(y)$ is a constant.

For the principle of conservation of energy, since the particle is frictionless,

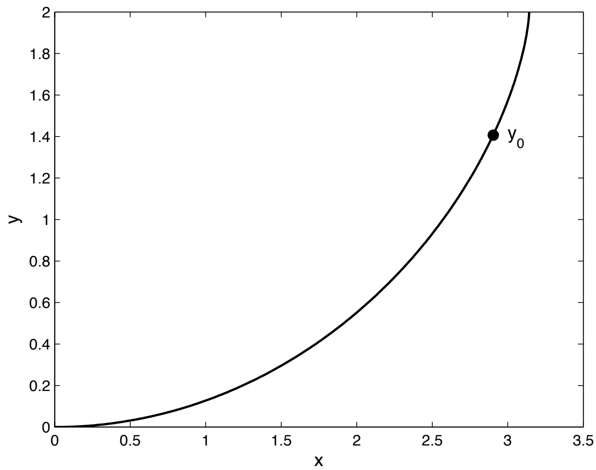

Figure 3: Tautochrone curve

and thus loses no energy to heat, its kinetic energy at any point is exactly equal to the difference in potential energy from its starting point. Formally:

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mg(y_0 - y)$$

where $\frac{ds}{dt}$ is the velocity of the particle, s the distance measured along the curve and $mg(y_0-y)$ the gravitational potential energy gained in falling from an initial height y_0 to a height y.

$$\begin{aligned} \frac{ds}{dt} &= \pm\sqrt{2g(y_0 - y)} \\ dt &= \pm\frac{ds}{\sqrt{2g(y_0 - y)}} \\ dt &= -\frac{1}{\sqrt{2g(y_0 - y)}}\frac{ds}{dy}dy \end{aligned}$$

In the last equation, we've anticipated writing the distance remaining along the curve as a function of height ($s(y)$) and recognized that the distance remaining must decrease as time increases (thus the minus sign).
Now we integrate from $y = y_0$ to $y = 0$ to get the total time required for the particle to fall:

$$T(y_0) = \int_{y=y_0}^{y=0} dt = \frac{1}{\sqrt{2g}} \int_0^{y_0} \frac{1}{\sqrt{y_0 - y}} \frac{ds}{dy}\, dy$$

This is called **Abel's integral equation** and allows us to compute the total time required for a particle to fall along a given curve (for which $\frac{ds}{dy}$ would be easy to calculate). Abel's mechanical problem is the opposite one: given $T(y_0)$ we wish to find $\frac{ds}{dy}$, from which an equation for the curve would follow in a straightforward manner.
To proceed, we note that the integral on the right is the convolution of $\frac{ds}{dy}$ with $\frac{1}{\sqrt{y}}$ and thus take the *Laplace transform* of both sides:

$$\mathcal{L}[T(y_0)] = \frac{1}{\sqrt{2g}} \mathcal{L}\left[\frac{1}{\sqrt{y}}\right] \mathcal{L}\left[\frac{ds}{dy}\right]$$

Since $\mathcal{L}\left[\frac{1}{\sqrt{y}}\right] = \sqrt{\pi} z^{-\frac{1}{2}}$, we now have an expression for the Laplace transform of $\frac{ds}{dy}$ in terms of $T(y_0)$'s Laplace transform:

$$\mathcal{L}\left[\frac{ds}{dy}\right] = \sqrt{\frac{2g}{\pi}} z^{\frac{1}{2}} \mathcal{L}[T(y_0)]$$

For the tautochrone problem, $T(y_0) = T_0$ is constant. Since the Laplace transform of 1 is $\frac{1}{z}$, we proceed:

$$\mathcal{L}\left[\frac{ds}{dy}\right] = \sqrt{\frac{2g}{\pi}} T_0 z^{-\frac{1}{2}}$$

Making use again of the Laplace transform above, we anti-transform and conclude:

$$\frac{ds}{dy} = T_0 \frac{\sqrt{2g}}{\pi} \frac{1}{\sqrt{y}}.$$

Now if we let

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \qquad \text{and} \qquad b = \frac{2gt^2}{\pi^2}$$

we obtain

$$\frac{dx}{dy} = \sqrt{\frac{b-y}{y}} \qquad \text{from which} \qquad x = \int \sqrt{\frac{b-y}{y}}\, dy + c$$

with the substitutions

$$y = b\sin^2\phi, \quad b - y = b\cos^2\phi \quad b = \frac{R}{2} \quad \text{and} \quad dy = 2b\sin\cos\phi d\phi$$

$$x = \int \sqrt{\frac{R\cos^2\phi}{R\sin^2\phi}}\, R\sin\cos\phi\, d\phi = R\int \cos^2\phi\, d\phi + c$$

Recalling

$$\cos^2\phi = \frac{1+\cos 2\phi}{2} \qquad \text{with} \qquad 2\phi = \theta$$

we finally have

$$\begin{cases} x = R(\theta \pm \sin\theta) \\ y = R(1-\cos\theta) \end{cases} .$$

References

[1] R. Gorenflo, S. Vessella, *Abel Integral Equation* - Analysis and Application, Springer-Verlag Berlin Heidelber, 1991

[2] F. Mainardi, Lecture Notes on Mathematical Physics